

Categorifications from planar diagrammatics

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August 30, 2010

Abstract

A diagrammatic presentation of functors and natural transformations and the virtues of biadjointness are discussed. We then review a graphical description of the category of Soergel bimodules and a diagrammatic categorification of positive halves of quantum groups. These notes are a write-up of Takagi lectures given by the author in Hokkaido University in June 2009.

1 Planarity of biadjointness

Adjoint functors, since their discovery by Daniel Kan [22] in 1958, have become quite ubiquitous in mathematics, with their universality well-documented already in the Wikipedia. We hope that biadjoint functors, which are pairs of functors (E, F) such that F is both right and left adjoint of E , will prove to be of importance as well.

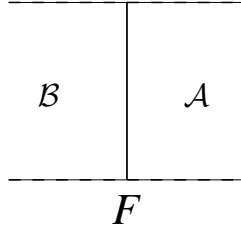
Let us begin by reviewing the topological meaning of adjointness and biadjointness. We will depict a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ by a mark on a horizontal interval, with the half-intervals to the right and left of the mark labelled by \mathcal{A} and \mathcal{B} , respectively.

$$\begin{array}{ccccc} \mathcal{B} & & F & & \mathcal{A} \\ \text{-----} & & | & & \text{-----} \end{array}$$

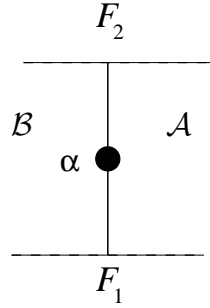
Composition $F_n \dots F_1 : \mathcal{A}_1 \longrightarrow \mathcal{A}_{n+1}$ of functors $F_i : \mathcal{A}_i \longrightarrow \mathcal{A}_{i+1}$ is depicted by placing marks for F_n, \dots, F_1 in a row, with the intervals labelled by categories $\mathcal{A}_1, \dots, \mathcal{A}_{n+1}$ reading from right to left.

$$\begin{array}{ccccccc} \mathcal{A}_{n+1} & & F_n & & \mathcal{A}_n & & & & \mathcal{A}_3 & & F_2 & & \mathcal{A}_2 & & F_1 & & \mathcal{A}_1 \\ \text{-----} & & | & & \text{-----} & & & & \text{-----} & & | & & \text{-----} & & | & & \text{-----} \end{array}$$

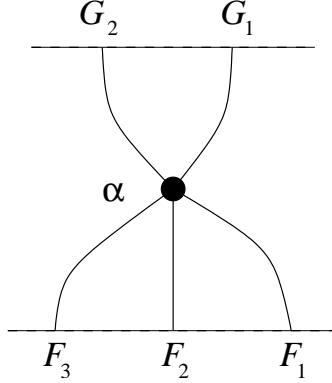
The identity natural transformation $1_F : F \Rightarrow F$ is depicted by a vertical interval drawn in a rectangular region of the plane, with the two areas labelled by categories \mathcal{A} and \mathcal{B} .



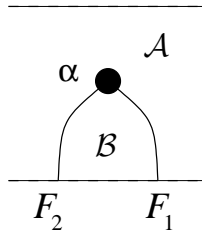
A natural transformation $\alpha : F_1 \Rightarrow F_2$, where F_1, F_2 are functors from \mathcal{A} to \mathcal{B} , is depicted by a dot on a vertical line.



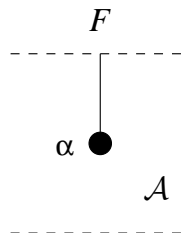
A natural transformation $\alpha : F_n \dots F_1 \rightarrow G_m \dots G_1$ can be depicted by merging lines for the identity transformations of F_n, \dots, F_1 into a point and then splitting it into lines for the identities of G_m, \dots, G_1 .



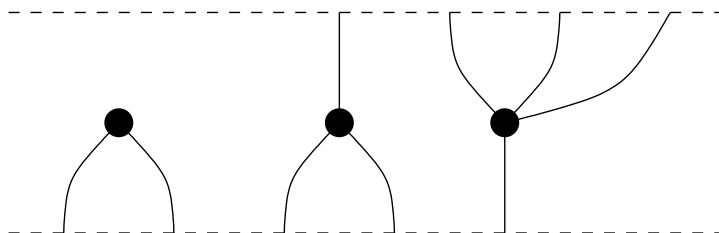
It is useful not to label the identity functor $\text{Id}_{\mathcal{A}}$ by anything and instead depict it by a horizontal interval labelled \mathcal{A} . Likewise, the identity natural transformation of this functor is denoted by a region labelled \mathcal{A} . With these rules, we can depict $\alpha : F_2 F_1 \Rightarrow \text{Id}_{\mathcal{A}}$ by the following picture.



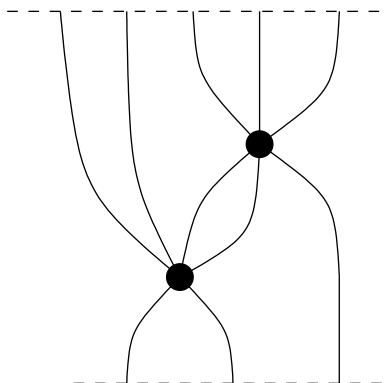
Here \mathcal{B} is the target category for functor F_1 and the source category for functor F_2 . Similarly, below is a picture for $\alpha : \text{Id}_{\mathcal{A}} \Rightarrow F$, where F is an endofunctor of \mathcal{A} .



The two possible types of compositions of natural transformations are depicted by either placing diagram in parallel or stacking them vertically; the next picture is an example of horizontal composition of three compatible natural transformations.



An example of vertical composition of two natural transformations, with some identity transformations thrown in, is given below.



It is important that strands always go “up”, that is, strands are not yet allowed to have U-turns. Isotopies of strands are allowed, as long as they don’t create U-turns. On some very informal level, these diagrams are analogous to planar projections of braids (braids don’t have U-turns either).

Thus, 2-dimensional planar pictures denote natural transformations, with regions of the picture labelled by categories, strands by functors and nodes by natural transformations.

A planar diagram without boundary points (a closed diagram) determines an endomorphism of $\text{Id}_{\mathcal{A}}$, where \mathcal{A} is the label of the outside region, i.e., an element of the center of category \mathcal{A} .

This setup, sometimes called string notation for 2-categories, is Poincare dual to the more common one where categories are depicted by points, functors by arrows and natural transformations by 2-cells. The same setup can be used with any 2-category in lieu of the 2-category of natural transformations. Regions of the diagrams will be labelled by objects of the 2-category, edges by 1-morphisms, and nodes by 2-morphisms.

A monoidal category is a 2-category with a single object. In this case we can avoid labelling regions, denote tensor product of objects of a monoidal category by a sequence of marks on a line and morphisms between tensor products by nodes with source and target arrows (there is a standard way of dealing with the case when the monoidal category is not strict; we omit the details). This notation is common in the diagrammatics for the representation categories of simple Lie algebras and quantum groups.

We now come to adjointness. Functors $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ are adjoint if there are isomorphisms

$$\text{Hom}_{\mathcal{B}}(FX, Y) \cong \text{Hom}_{\mathcal{A}}(X, GY)$$

functorial in $X \in \mathcal{A}$ and $Y \in \mathcal{B}$. An adjunction is equivalent to having the unit and the counit

$$\eta : \text{Id}_{\mathcal{A}} \Rightarrow GF, \quad \epsilon : FG \Rightarrow \text{Id}_{\mathcal{B}}$$

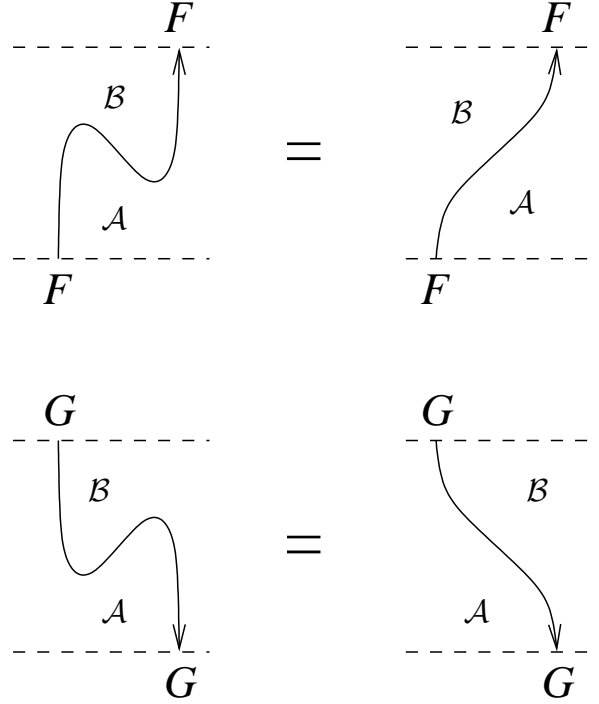
natural transformations that satisfy the following equations:

- natural transformation $F \xrightarrow{1_F \circ \eta} FGF \xrightarrow{\epsilon \circ 1_F} F$ equals 1_F ,
- natural transformation $G \xrightarrow{\eta \circ 1_G} GFG \xrightarrow{1_G \circ \epsilon} G$ equals 1_G .

Let us depict η and ϵ by a cup and a cap diagram, respectively. To distinguish between functors F and G we equip the strands with up orientation near F and down orientation near G .



The equations turn into relations on planar diagrams



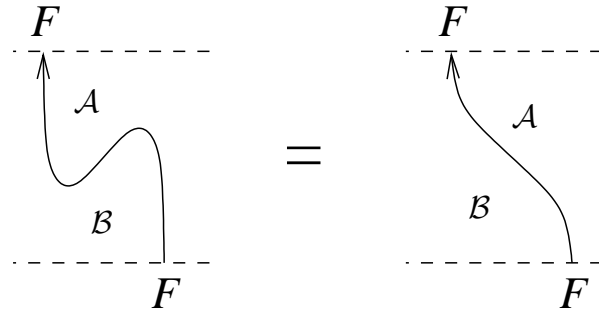
These relations have a natural interpretation via isotopies of arcs, but give us only two out of four basic isotopy relations on oriented arcs in the plane. To get complete isotopy invariance, we assume that F is also a right adjoint to G and the natural transformations describing the second adjointness are fixed

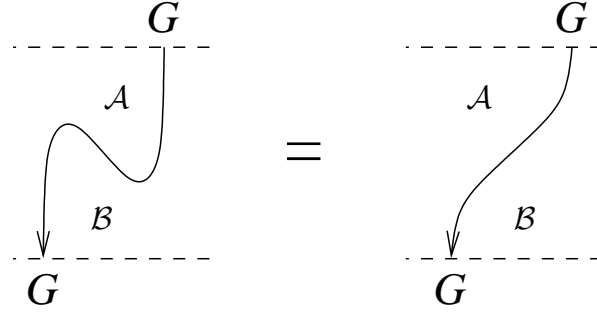
$$\underline{\eta} : \text{Id}_{\mathcal{B}} \Rightarrow FG, \quad \underline{\epsilon} : GF \Rightarrow \text{Id}_{\mathcal{A}}.$$

They satisfy the relations

- natural transformation $F \xrightarrow{\eta \circ 1_F} FGF \xrightarrow{1_F \circ \epsilon} F$ equals 1_F ,
- natural transformation $G \xrightarrow{1_G \circ \eta} GFG \xrightarrow{\epsilon \circ 1_G} G$ equals 1_G ,

whose graphical interpretation is that of the two remaining types of arc isotopies:

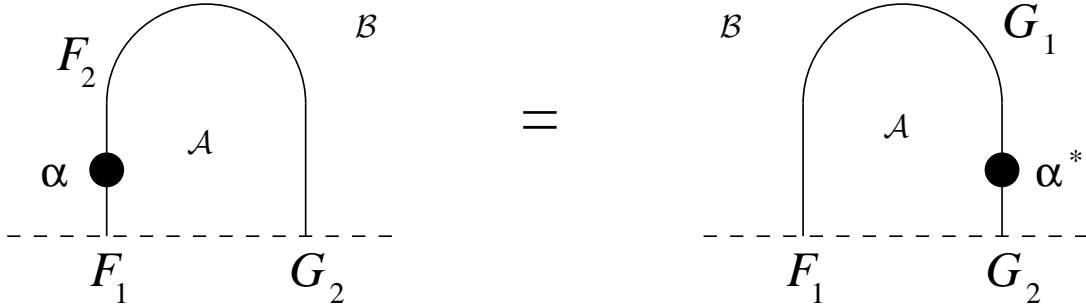




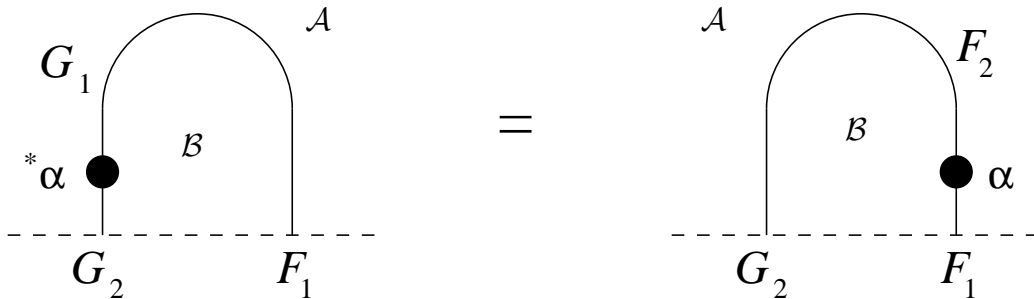
Thus, given a biadjoint pair (F, G) we can make sense out of any diagram built from the four possible (F, G) -cups and caps. We say that G is a biadjoint of F and F a biadjoint of G . A functor admitting a biadjoint is also called a *Frobenius* functor, and a biadjoint pair (F, G) is called a *Frobenius pair*. Yet another terminology for a biadjoint pair is *ambidextrous adjunction* [31]. One of the first interesting examples of biadjoint pairs that appeared in mathematics (specifically, in the modular representation theory) were induction and restriction functors for inclusions of finite groups. For a simpler example, take any invertible functor: its inverse is its biadjoint.

One often works with many pairs of biadjoint functors, which requires compatibility, namely given biadjunctions (F_1, G_1) and (F_2, G_2) such that $F_2 F_1$ is defined, the biadjunction for $(F_2 F_1, G_1 G_2)$ should be built in the obvious diagrammatic way via the composition rules.

Biadjointness allows us to move dots on strands through cups and caps. Say, we have a transformation $\alpha : F_1 \Rightarrow F_2$ and biadjoint pairs $(F_1, G_1), (F_2, G_2)$. There exists a unique transformation $\alpha^* : G_2 \Rightarrow G_1$ that satisfies the equality



${}^*\alpha : G_2 \Rightarrow G_1$ is defined similarly, via the equality



From the topological viewpoint, it is convenient to require $^*\alpha = \alpha^*$ for all natural transformations α between our pairs of biadjoint functors, for then α always turns into α^* or α no matter what sequence of caps and cups it goes through. We refer to this property as *cyclic biadjointness* (also called *even-handed structure* in [3]).

With these assumptions, there is now a complete isotopy invariance for the string diagrams. Functors for these diagrams are selected from a collection of cyclic biadjoints that satisfy the above compatibility condition for their compositions. Any natural transformation between compositions of these functors can potentially be depicted. Some are glued out of the basic ones, others require introducing new nodes with multiple input and output strands. Once a new node for a natural transformation between composition appears, it may be used as a building block for more complicated planar diagrams. Due to cyclicity, these nodes can be isotoped in the plane.

Remark: If the cyclic order of functors around the node has a rotational symmetry, it might not extend to a symmetry of the node (but in some natural examples it does). For instance, given a node for $\alpha : F \Rightarrow F$ where (F, F) is a biadjoint pair (of course, $F : \mathcal{A} \longrightarrow \mathcal{A}$ is then an endofunctor), $\alpha^* : F \Rightarrow F$ does not have to be equal to α .

The above discussion generalizes, allowing us to depict elements of strict 2-categories with suitable duality properties, mirroring those of cyclic biadjointness.

The planar interpretation of biadjointness has been a folklore for a number of years and appeared in [40, 12]. String notation, in the case of 2-categories with one object (monoidal categories), was introduced and made rigorous by Joyal and Street [21]. Nowadays, it can be found on YouTube, in a series of videos "String diagrams" by TheCatsters.

2 Biadjointness in topology and algebra

Planar diagrams of lines labelled by functors and natural transformations can be thought of as suitably decorated 1-dimensional cobordisms embedded in \mathbb{R}^2 . There is also a direct relation of biadjoint functors to cobordisms in all dimensions. Let Cob_{n+2} be the 2-category of $(n+2)$ -dimensional smooth cobordisms with boundary and corners. Closed n -dimensional manifolds K are objects of Cob_{n+2} , while an $(n+1)$ -dimensional cobordism M is a 1-morphism from object $\partial_0 M$ to object $\partial_1 M$. An $(n+2)$ -dimensional cobordism N with corners is a 2-morphism from $\partial_0 N$ to $\partial_1 N$. Here $\partial_0 N, \partial_1 N$ are $(n+1)$ -manifolds with boundary, and the boundary of N consists of 4 pieces: $\partial_0 N, \partial_1 N$, and products $(\partial_0 \partial_0 N) \times [0, 1]$, $(\partial_1 \partial_1 N) \times [0, 1]$. Corners of N are four n -manifolds $\partial_i \partial_j N$, $i, j \in \{0, 1\}$.

For each 1-morphism M there is the reverse 1-morphism $r(M)$ from $\partial_1 M$ to $\partial_0 M$ given by flipping M . We claim that $(M, r(M))$ constitutes a cyclic biadjoint pair. Multiply M by an interval and then fan out the resulting $(n+2)$ -manifold with corners so that it gives two $(n+2)$ -cobordisms between $r(M)M$ and $\text{Id}_{\partial_0 M}$ (one in each direction) and two $(n+2)$ -cobordisms between $Mr(M)$ and $\text{Id}_{\partial_1 M}$. The relations on these four

cobordisms are exactly the same as those satisfied by natural transformations $\eta, \epsilon, \bar{\eta}, \bar{\epsilon}$ of a biadjoint pair (for more details and pictures see [24, Section 6.3]). In particular, given a 2-functor F from Cob_{n+2} into the 2-category of categories, functors, and natural transformations, the functors $F(M)$ and $F(r(M))$ are canonically biadjoint.

If the target of F is the 2-category of additive categories and additive functors, we say that F is an $(n+2)$ -dimensional TQFT with corners (or extended TQFT). Most of the time we require manifolds to be oriented, often additionally decorated by some structure, such as a spin structure, and sometimes place restrictions on topological type of manifolds and cobordisms. There are interesting mathematically understood examples in dimensions $n+2 = 2, 3, 4$. In dimension 2, they come from commutative Frobenius algebras. In dimension 3, the Witten-Reshetikhin-Turaev TQFT and its relatives emerge from Chern-Simons theory and representation theory of quantum groups, while the Rozansky-Witten TQFT comes from the derived category of coherent sheaves on holomorphic symplectic manifolds. Donaldson-Floer, Seiberg-Witten and Heegaard Floer theory (all closely interrelated) provide famous examples of 4D TQFTs.

Any extended TQFT produces an abundance of biadjoint pairs, namely one pair $(F(M), F(r(M)))$ for each $(n+1)$ -cobordism M . These biadjoint functors go between categories $F(K)$ assigned to closed n -manifolds K . When searching for TQFTs, it is useful to look for collections of categories that admit many biadjoint functors between each other. Examples of such collections include:

- Derived categories of coherent sheaves on a Calabi-Yau variety. Any sheaf on the product $X \times Y$ induces a pair of convolution functors between $D^b(X)$ and $D^b(Y)$. When $\dim(X) = \dim(Y)$, this is a biadjoint pair. When dimensions don't match, the functors are almost biadjoint (biadjoint up to a shift in the derived category).
- Categories of modules over finite-dimensional symmetric algebras and their derived categories. The functor of tensoring with a finitely-generated bimodule over symmetric algebras which is left and right projective has a bijoint.
- Fukaya-Floer categories of symplectic manifolds.
- Derived categories of sheaves on flag varieties.
- Various categories that appear in categorification of representations of Hecke algebras and quantum groups. For instance, functor \mathcal{E}_i that categorifies generator E_i of a simple Lie algebra/quantum group is biadjoint (or almost biadjoint) to the functor \mathcal{F}_i categorifying generator F_i , see [13, 32, 28, 44]. Some of the earliest examples of categorifications of Hecke algebra and $\mathfrak{sl}(k)$ representations [19, 5, 46, 39] came from highest weight categories of representations of $\mathfrak{sl}(n)$, with E_i and F_i being translation functors (direct summands of tensor products with finite-dimensional modules) or Zuckerman functors. Biadjointness of translation functors was used already in [6]. In Ariki's categorification [1] of irreducible $\mathfrak{sl}(k)$ and affine $\mathfrak{sl}(k)$ representations, functors \mathcal{E}_i and \mathcal{F}_i are biadjoint

as well, being induction and restriction functors between Ariki-Koike-Cherednik cyclotomic quotients of affine Hecke algebras.

The issue of biadjointness in the Reshetikhin-Turaev-Witten TQFT is not prominent, since the category associated to the circle is semisimple \mathbb{C} -linear and functors associated to surfaces with boundary are \mathbb{C} -linear as well. Such functors are guaranteed to have biadjoints, which can be described in a simple combinatorial way. In contrast, the category associated to the circle in the Rozansky-Witten TQFT is a version of the derived category of coherent sheaves on a holomorphic symplectic manifold X , which is necessarily Calabi-Yau, so that one of the examples on the above list becomes relevant. Triangulated categories behind extended Donaldson-Floer and Ozsváth-Szabó theories [34] are assigned to surfaces and closely related to Fukaya-Floer categories of the representation variety of the fundamental group of a surface and of the symmetric power of a surface, respectively. Extended 4D TQFT that controls Ozsváth-Szabó 3-manifold homology and 4-manifold invariants is being unravelled by Lipshitz, Ozsváth and Thurston [34].

We list a few nice features of biadjoint pairs:

- A biadjoint functor commutes with both limits and colimits.
- If a functor F between additive categories has left or right adjoint, it is additive. In particular, in biadjoint pairs (F, G) between additive categories both F and G are additive functors.
- In a biadjoint pair (F, G) of functors between abelian categories both F and G are additive and exact, take projectives to projectives and injectives to injectives.

To an abelian category \mathcal{A} there is assigned its Grothendieck group $G_0(\mathcal{A})$, an abelian group with generators given by symbols $[M]$ of objects of \mathcal{A} and defining relations $[M] = [M'] + [M'']$ for each short exact sequence

$$0 \longrightarrow M'' \longrightarrow M \longrightarrow M' \longrightarrow 0.$$

Often, it also makes sense to consider the group $K_0(\mathcal{A})$ generated by symbols of projective objects $[P]$ modulo relations $[P] = [P'] + [P'']$ if $P \cong P' \oplus P''$. The obvious homomorphism $\phi_{\mathcal{A}} : K_0(\mathcal{A}) \longrightarrow G_0(\mathcal{A})$ is, in general, neither injective nor surjective. The homomorphism takes $[P]$ to $[P]$, so this symbol notation might be ambiguous on projectives.

A biadjoint pair (F, G) induces homomorphisms

$$\begin{aligned} [F] : K_0(\mathcal{A}) &\longrightarrow K_0(\mathcal{B}), & G_0(\mathcal{A}) &\longrightarrow G_0(\mathcal{B}), \\ [G] : K_0(\mathcal{B}) &\longrightarrow K_0(\mathcal{A}), & G_0(\mathcal{B}) &\longrightarrow G_0(\mathcal{A}) \end{aligned}$$

that commute with $\phi_{\mathcal{A}}, \phi_{\mathcal{B}}$. These homomorphisms take $[M]$ to $[FM]$, respectively $[GM]$. In contrast, an exact functor F would induce a homomorphism $G_0(\mathcal{A}) \rightarrow G_0(\mathcal{B})$ but not necessarily a homomorphism between K_0 's, since it might not take projectives to projectives. Likewise, a functor taking projectives to projectives would induce a homomorphism on K_0 's but not on G_0 's. Biadjoint pairs between abelian categories give the best behaving functors from this perspective. They can be thought of as categorifying pairs of adjoint operators. Namely, assume that \mathcal{A}, \mathcal{B} are \mathbb{k} -linear, over a field \mathbb{k} , and hom spaces between objects are finite-dimensional over \mathbb{k} . This gives bilinear forms on $K_0(\mathcal{A}), K_0(\mathcal{B})$ determined by

$$([P], [Q])_{\mathcal{A}} := \dim \operatorname{Hom}_{\mathcal{A}}(P, Q)$$

for projectives $P, Q \in \mathcal{A}$, likewise for \mathcal{B} . Adjointness isomorphisms

$$\operatorname{Hom}_{\mathcal{B}}(FP, Q) \cong \operatorname{Hom}_{\mathcal{A}}(P, GQ), \quad \operatorname{Hom}_{\mathcal{B}}(Q, FP) \cong \operatorname{Hom}_{\mathcal{A}}(GQ, P)$$

descend to relations

$$([F]v, w)_{\mathcal{B}} = (v, [G]w)_{\mathcal{A}}, \quad (w, [F]v)_{\mathcal{B}} = ([G]w, v)_{\mathcal{A}}, \quad v \in K_0(\mathcal{A}), w \in K_0(\mathcal{B}),$$

showing that $[F]$ and $[G]$ become adjoint operators on real vector spaces $K_0(\mathcal{A}) \otimes \mathbb{R}$ and $K_0(\mathcal{B}) \otimes \mathbb{R}$ relative to these two bilinear forms (in interesting examples the forms are often symmetric and nondegenerate).

3 Diagrammatics for Soergel bimodules

The Iwahori-Hecke algebra H_n of the symmetric group S_n has generators T_i , $1 \leq i \leq n-1$, and defining relations

$$\begin{aligned} T_i^2 &= (q-1)T_i + q, \\ T_i T_j &= T_j T_i \quad \text{for } |i-j| \geq 2, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \end{aligned}$$

where q is a formal parameter. In present-day mathematics it appears in two seemingly different ways:

I) As a finite-dimensional quotient of the group algebra of the braid group, providing invariants of braid and links, when the latter are realized as closures of braids. Two braid closures produce the same link if they are related by a finite sequence of Markov moves. The algebraic counterpart of the closure operation is taking trace of an operator. The Ocneanu trace on the Hecke algebra behaves well under the Markov moves and can be normalized to produce an invariant of links known as the HOMFLY-PT polynomial (named with the initials of eight people who independently discovered it).

More algebraically, the Hecke algebra is the commutant of the action of the quantum group $U_q(sl(k))$ on $V^{\otimes n}$ where $\dim(V) = k$ and $k \geq n$.

II) As the endomorphism algebra of the representation of $GL(n, \mathbb{F}_q)$ induced from the trivial representation of the subgroup \mathcal{B} of all invertible upper-triangular matrices. Here $GL(n, \mathbb{F}_q)$ is the finite group of invertible $n \times n$ matrices with coefficients in the finite field \mathbb{F}_q with q elements, hence in this interpretation q is no longer a formal variable. Case $q = 1$ also fits into this framework, corresponding to $GL(n)$ over the one-element field, which is the symmetric group (and the Iwahori-Hecke algebra specializes to the group algebra of the symmetric group when $q = 1$).

Construction I) of the Hecke algebra is fundamental for low-dimensional topology, construction II) and its generalizations is indispensable in representation theory. The second construction also leads to a categorification of the Iwahori-Hecke algebra. The functions on the finite set $GL(n, \mathbb{F}_q)/\mathcal{B}$ become sheaves on the flag variety $GL(n)/\mathcal{B}$, either in the étale topology over a finite field, or sheaves of \mathbb{C} -vector spaces on the flag variety over the field \mathbb{C} . Categorification of H_n was constructed by Soergel [45], who also stated it in a sheaf-free language (which works for arbitrary Weyl groups). We describe it here in the symmetric group case.

It is convenient to introduce $t = \sqrt{q}$ and view H_n as a $\mathbb{Z}[t, t^{-1}]$ -algebra. Then H_n is also generated by $b_i = t^{-1}(T_i + 1)$, $1 \leq i \leq n - 1$, with defining relations

$$b_i^2 = (t + t^{-1})b_i \quad (1)$$

$$b_i b_j = b_j b_i \text{ for } |i - j| \geq 2 \quad (2)$$

$$b_i b_{i+1} b_i + b_{i+1} = b_{i+1} b_i b_{i+1} + b_i. \quad (3)$$

Soergel's categorification of H_n is built out of bimodules over the polynomial ring $R = \mathbb{C}[x_1, \dots, x_n]$. The ring is graded, $\deg(x_i) = 2$, and all the bimodules are graded as well. Grading shift $\{1\}$ up by one is an automorphism of the category of graded bimodules. For each $1 \leq i \leq n - 1$ let $R^i \subset R$ be the subring that consists of polynomials invariant under the transposition $x_i \leftrightarrow x_{i+1}$. Then $R \otimes_{R^i} R$ is a graded R -bimodule and we let

$$B_i := R \otimes_{R^i} R\{-1\}.$$

Form the category \mathcal{SC}_1 whose objects are tensor products of B_i 's and morphisms are homomorphisms of bimodules. By adding finite direct sums, grading shifts, restricting to grading-preserving homomorphisms, and forming the Karoubi envelope, one arrives at the category \mathcal{SC} . We call the objects of this category Soergel bimodules. One of Soergel's results is that \mathcal{SC} categorifies the Hecke algebra H_n , i.e. the Grothendieck ring of \mathcal{SC} is canonically isomorphic to H_n ,

$$K_0(\mathcal{SC}) \cong H_n,$$

with $[B_i]$, the symbol of B_i , going to b_i under the isomorphism. Grading shift corresponds to multiplication by t : $[M\{1\}] = t[M]$. Multiplication in H_n corresponds to the

tensor product of bimodules,

$$[M] \cdot [N] := [M \otimes_R N].$$

Indecomposables in \mathcal{SC} are in a bijection, up to grading shift, with elements w of the symmetric group. Bimodule $B_e = R$, where e is the trivial permutation, bimodules $B_{s_i} = B_i$, where $s_i = (i, i+1)$, and, inductively, if $l(ww') = l(w) + l(w')$, $B_{ww'}$ is the only indecomposable summand of $B_w \otimes_R B_{w'}$ that is not isomorphic, up to a grading shift, to B_u for some u with $l(u) < l(w) + l(w')$. Equivalently, B_w is determined by the condition that it appears as a direct summand of $B_{\underline{i}} := B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_d}$, where $\underline{i} = i_1 \dots i_d$ and $s_{i_1} \dots s_{i_d}$ is a reduced presentation of w , and does not appear as direct summand of any $B_{\underline{i}}$, for sequences \underline{i} of length less than $d = l(w)$.

Defining relations on b_i 's become isomorphisms of graded bimodules

$$B_i \otimes B_i \cong B_i\{1\} \oplus B_i\{-1\}, \quad (4)$$

$$B_i \otimes B_j \cong B_j \otimes B_i \text{ for } |i - j| \geq 2, \quad (5)$$

$$(B_i \otimes B_{i+1} \otimes B_i) \oplus B_{i+1} \cong (B_{i+1} \otimes B_i \otimes B_{i+1}) \oplus B_i. \quad (6)$$

The last isomorphism comes from decompositions

$$B_i \otimes B_{i+1} \otimes B_i \cong B_i \oplus (R \otimes_{i,i+1} R\{-3\}) \quad (7)$$

$$B_{i+1} \otimes B_i \otimes B_{i+1} \cong B_{i+1} \oplus (R \otimes_{i,i+1} R\{-3\}). \quad (8)$$

Here $R \otimes_{i,i+1} R$ is the tensor product of two R 's over the subring of S_3 -invariants, with the action of S_3 by permutations of x_i, x_{i+1}, x_{i+2} .

When $n = 3$, the Soergel category has 6 indecomposables, up to grading shifts. They are

$$R, \quad B_1, \quad B_2, \quad B_1 \otimes_R B_2, \quad B_2 \otimes_R B_1, \quad R \otimes_{i,i+1} R\{-3\}.$$

Their images in the Grothendieck ring $K_0(\mathcal{SC})$ are

$$1, \quad b_1, \quad b_2, \quad b_1 b_2, \quad b_2 b_1, \quad b_1 b_2 b_1 - b_1 = b_2 b_1 b_2 - b_2.$$

N. Libedinsky [33] presented \mathcal{SC} in the $n = 2$ case (and, more generally, in the so-called right-angled case, that we don't discuss) via generators and relations on morphisms. We now explain, following [15], a generalization of his presentation to an arbitrary n . This approach provides a graphical description of homomorphisms between tensor products of Soergel bimodules.

Tensoring with a Soergel bimodule is an endofunctor in the category of (graded) R -modules and homomorphisms between tensor products corresponds to natural transformations of functors. Start with the simplest Soergel bimodule R . The ring R is commutative and multiplication by any element of R is an endomorphism of the identity functor (tensoring with R), thus belongs to the center of the category of R -modules. We denote by a box labelled i the multiplication by the generator x_i of R .

$$\boxed{i}$$

One can think of this box as freely floating in a region. Boxes can float past each other in any direction (relative height change isotopy corresponds to commutativity of central elements). A collection of floating boxes denotes the product of corresponding generators. Any element of R is a \mathbb{C} -linear combinations of products of boxes.

A vertical line labelled i will denote the identity endomorphism of the bimodule B_i . An important feature of B_i is that it is selfadjoint (in the category of Soergel bimodules), i.e. left and right adjoint to itself, and this selfadjointness is cyclic. Therefore, we can introduce unoriented cup and cap diagrams labelled by i to denote units and counits of biadjunctions (in pictures below, label i is omitted). These diagrams have zero degree.

Moreover, B_i is a Frobenius object. Namely, there are homomorphisms

$$R \longrightarrow B_i, \quad B_i \longrightarrow R, \quad B_i \otimes B_i \longrightarrow B_i, \quad B_i \longrightarrow B_i \otimes B_i$$

of degrees $1, 1, -1, -1$ respectively, that we depict by

$$\begin{array}{c} \downarrow \quad \uparrow \quad \text{Y-junction} \quad \text{X-junction} \end{array} \quad (9)$$

and that satisfy the axioms of a Frobenius algebra object:

$$\begin{array}{c} \text{Associativity diagrams for multiplication and comultiplication} \end{array} \quad (10)$$

$$\begin{array}{c} \text{Frobenius axiom diagrams} \end{array} \quad (11)$$

$$\begin{array}{c} \text{Isotopy diagrams for cups and caps} \end{array} \quad (12)$$

$$\begin{array}{c} \text{Counit and unit diagrams} \end{array} \quad (13)$$

$$\begin{array}{ccc}
\text{concave curve} & = & \text{Y-junction with dot at bottom} \\
\text{convex curve} & = & \text{Y-junction with dot at top}
\end{array} \tag{14}$$

We list other relations that involve strands of one color only:

$$\boxed{i} \left| + \right| \boxed{i+1} = \left| \boxed{i} + \right| \boxed{i+1} \tag{15}$$

$$\boxed{i} \boxed{i+1} \left| \right| = \left| \boxed{i} \boxed{i+1} \right| \tag{16}$$

$$\boxed{j} \left| \right| = \left| \boxed{j} \right|, \quad |i-j| > 1. \tag{17}$$

These come directly from the definition of B_i . Furthermore, we impose

$$\begin{array}{ccc}
\text{vertical line with two dots} & = & \boxed{i} \left| - \right| \boxed{i+1} \\
\text{vertical line with two dots (reversed)} & = & \boxed{i} - \boxed{i+1}
\end{array} \tag{18}$$

$$\text{loop with a tail} = 0 \tag{19}$$

These relations imply, in particular,

$$\begin{aligned}
& \text{Two parallel vertical lines} = \text{Two vertical lines with a small black square on each} = \\
& \text{Diagram with a horizontal line connecting two vertices labeled } i \text{ minus Diagram with a horizontal line connecting two vertices labeled } i+1 \\
& = \text{Diagram with a vertical line connecting two vertices labeled } i \text{ minus Diagram with a vertical line connecting two vertices labeled } i+1
\end{aligned} \tag{20}$$

leading to the decomposition $B_i \otimes B_i \cong B_i\{1\} \oplus B_i\{-1\}$.

We now describe generators and relations for interactions of two adjacent colors i and $i + 1$. Below, thin lines represent B_i and thick lines B_{i+1} . The six-valent vertex denotes the composition

$$B_i \otimes B_{i+1} \otimes B_i \longrightarrow R_{\otimes_{i,i+1}} R\{-3\} \longrightarrow B_{i+1} \otimes B_i \otimes B_{i+1}$$

of projection from the tensor product of 3 bimodules onto its most interesting direct summand and inclusion of this summand into the other triple tensor product. It turns out that the 6-valent vertex possesses full rotational invariance:

$$\text{Diagram with three thick lines and three thin lines} = \text{Diagram with three thick lines and three thin lines} = \text{Diagram with three thick lines and three thin lines} \tag{21}$$

Direct sum decompositions of $B_{i,i+1,i}$ and $B_{i+1,i,i+1}$ are encoded by the relation

$$\text{Three vertical lines} = \text{Diagram with a vertical line connecting two vertices} - \text{Diagram with a vertical line connecting two vertices} \tag{22}$$

and its relative given by reversing the thickness of lines. Other relations are (add their reverses as well)

$$\begin{array}{c} \diagup \\ \times \\ \diagdown \end{array} = \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} + \begin{array}{c} \uparrow \\ \downarrow \\ \downarrow \end{array} \quad (23)$$

$$\begin{array}{c} \diagup \\ \times \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \times \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \times \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \times \\ \diagdown \end{array} \quad (24)$$

$$\begin{array}{c} \diagup \\ \times \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \times \\ \diagdown \end{array} \quad (25)$$

Lastly, i and j -colored lines are allowed to cross when $|i - j| > 1$. These crossings conclude the list of generating 2-morphisms. The following relations say that the crossings are essentially “virtual”, i.e. the lines freely go through each other (the solid line has color i and dashed – color j).

$$\begin{array}{c} \diagup \\ \times \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \times \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \times \\ \diagdown \end{array} \quad (26)$$

$$\begin{array}{c} \diagup \\ \times \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \times \\ \diagdown \end{array} \quad (27)$$

$$\begin{array}{c} \diagup \\ \times \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \times \\ \diagdown \end{array} \quad (28)$$

$$(29)$$

The next relation says that j -line interacts trivially with trivalent $i, i + 1$ -colored 6-vertex (necessarily $j < i - 1$ or $j > i + 2$).

$$(30)$$

The following relation shows trivial interaction between the crossings of i, j, k -colored lines (the colors are necessarily far apart $|i - j| > 1, |i - k| > 1, |j - k| > 1$).

$$(31)$$

Finally, the last and most sophisticated relation is an interaction between 6-valent vertices for $i, i + 1, i + 2$ (dotted line has color $i + 2$).

$$(32)$$

The main result of [15] is that the above generators and relations give a presentation of the Soergel category \mathcal{SC}_1 . We can say that \mathcal{SC}_1 is a finitely-presented \mathbb{C} -linear pivotal monoidal category. This presentation is manifestly planar (perhaps the term *planar category* can be used as a substitute for *pivotal monoidal category*). Of course, the self-adjointness of tensoring with B_i was a strong hint that \mathcal{SC}_1 should have a planar description.

Below we list some applications of this new viewpoint on the Soergel category.

(a) Planar presentation of \mathcal{SC}_1 leads to a new categorification [14] of the Temperley-Lieb algebra TL_n . The Temperley-Lieb algebra is the quotient of the Hecke algebra H_n by the relations $b_i b_{i \pm 1} b_i = b_i$. This algebra is fundamental for the construction of

the Jones polynomial and admits a categorification via bimodules over certain rings, see [24]. To categorify the relation $b_i b_{i\pm 1} b_i = b_i$ observe that after categorification both sides become Soergel bimodules $B_{i,i\pm 1,i}$ and B_i , and that the latter bimodule is a summand of the former. Thus, the equality can be categorified by setting the complementary summand to 0. Since the 6-valent vertex (21) is a map which goes through this summand, we simply set all 6-valent vertices to 0 and form the quotient monoidal category. A result of B. Elias [14] says that the Grothendieck ring of the quotient category is naturally isomorphic to (a $\mathbb{Z}[q, q^{-1}]$ -form) of the Temperley-Lieb algebra.

In an earlier categorification of the Temperley-Lieb algebra [24], a basis of homomorphisms between bimodules categorifying products of b_i 's was given by 3-dimensional objects, namely suitable decorated surfaces with boundary and corners embedded in $[0, 1]^3$. In Elias's approach, a basis is given by a certain collection of planar diagrams representing morphisms in the quotient category. A comparison between the two categorifications gives an example of *dimensional encoding* or *dimensional reduction*, when 3-dimensional information is flattened onto 2D, see [38, 48].

(b) R. Rouquier [43] constructed a braid group action on the category of complexes of Soergel bimodules up to chain homotopies. The braid group generators σ_i correspond to complexes

$$0 \longrightarrow B_i\{1\} \longrightarrow R \longrightarrow 0$$

with the differential being the multiplication map $R \otimes_{R^i} R \longrightarrow R$ (counit of B_i in graphical notation). This braid group action extends to an action of the category of braid cobordisms [16, 30]. The action can be rethought in the above diagrammatical language [16], providing a link between 2-dimensional defining relations in \mathcal{SC}_1 and braid cobordisms (4-dimensional objects).

(c) Taking Hochschild homology of Rouquier complexes in a suitable way leads to homology groups that turn out to depend only on the closure of a braid [25]. The resulting homology theory is triply-graded, coincides with the one introduced in [29], and can be viewed as a categorification of the Ocneanu trace, thus a categorification of the HOMFLY-PT polynomial. We hope that planar diagrammatics will help to understand this largely mysterious homology theory and its generalization by Webster and Williamson [51].

It is natural to expect a possible relation between diagrammatics for \mathcal{SC}_1 (and more general planar categories and planar 2-categories) and 2D (topological) field theories. Biadjointness is implicit everywhere in 2D QFTs, since the categories that appear there, such as derived categories of coherent sheaves on 3D Calabi-Yau manifolds, Fukaya-Floer categories, and categories of matrix factorizations in LG models admit a wealth of biadjoint functors, that come up in a natural way, via convolutions with sheaves on products of a pair of Calabi-Yau's, via convolutions with Lagrangians in the product of symplectic manifolds, and via matrix factorizations with potentials $f(x) - g(y)$. Planar

diagrams for morphisms in \mathcal{SC}_1 (and planar diagrams for 2-morphisms in similar 2-categories) can perhaps be viewed as world sheets of 2D QFTs with defect lines and vertices, with regions of the diagram labelled by different target objects.

On a speculative note, we suggest a noncommutative geometry language for \mathcal{SC} . Recall that we represent Soergel bimodule $B_{\underline{i}}$ for a sequence $\underline{i} = i_1 \dots i_d$ by placing dots labelled i_1, \dots, i_d on a line. Let's imagine that this line with labeled dots is a path, and refer to $B_{\underline{i}}$ as a path as well. More generally, we call any object of \mathcal{SC} a path. Indecomposable objects B_w are *geodesics*. Given two objects M, N of \mathcal{SC} , we think of the graded vector space $\text{Hom}_{\mathcal{SC}}(M, N)$ as categorified quantized area of surfaces stretched between M and N . The fact that the grading of $\text{Hom}_{\mathcal{SC}}(M, N)$ is bounded from below is loosely analogous to the same property of energy.

4 Categorification of quantum groups

The first example of a categorification of a bialgebra is apparently due to L. Geissinger [17]. Standard inclusions of symmetric groups $S_n \times S_m \subset S_{n+m}$ give rise to induction and restriction functors

$$\begin{aligned} \text{Ind}_{n,m} &: \mathbb{C}[S_n]\text{-mod} \times \mathbb{C}[S_m]\text{-mod} \longrightarrow \mathbb{C}[S_{n+m}]\text{-mod}, \\ \text{Res}_{n,m} &: \mathbb{C}[S_{n+m}]\text{-mod} \longrightarrow \mathbb{C}[S_n \times S_m]\text{-mod} \end{aligned}$$

between categories of (finite-dimensional) modules over these group algebras. Summing over all $n, m \geq 0$ produces functors

$$\text{Ind} : \mathcal{S} \times \mathcal{S} \longrightarrow \mathcal{S} \quad \text{Res} : \mathcal{S} \longrightarrow \mathcal{S} \otimes \mathcal{S},$$

where

$$\mathcal{S} := \bigoplus_{n \geq 0} \mathbb{C}[S_n]\text{-mod}, \quad \mathcal{S} \otimes \mathcal{S} := \bigoplus_{n, m \geq 0} \mathbb{C}[S_n \times S_m]\text{-mod}$$

is the direct sum of representation categories of symmetric groups, respectively products of symmetric groups. Functors Ind and Res induce homomorphisms of Grothendieck groups

$$[\text{Ind}] : K_0(\mathcal{S}) \otimes K_0(\mathcal{S}) \longrightarrow K_0(\mathcal{S}), \quad [\text{Res}] : K_0(\mathcal{S}) \longrightarrow K_0(\mathcal{S}) \otimes K_0(\mathcal{S}).$$

Here

$$K_0(\mathcal{S}) := \bigoplus_{n \geq 0} K_0(\mathbb{C}[S_n]\text{-mod}),$$

and the Grothendieck group $K_0(\mathbb{C}[S_n]\text{-mod})$ is a free abelian group with a natural basis given by symbols of simple S_n modules L_λ , parametrized by partitions λ of n (the group algebra $\mathbb{C}[G]$ is semisimple for any finite group G , any module is projective, and isomorphism classes of simple modules give rise to a basis of K_0). It is natural to identify $K_0(\mathcal{S})$ with the ring of symmetric functions in countably many variables by taking $[L_\lambda]$ to the Schur function s_λ .

Proposition 1 *Maps $[\text{Ind}]$ and $[\text{Res}]$ turn $K_0(\mathcal{S})$ into a biring.*

A biring is a bialgebra over \mathbb{Z} . This result, due to Geissinger [17], also tells us that the ring of symmetric functions has a natural comultiplication. The condition that comultiplication $\Delta = [\text{Res}]$ is a homomorphism, $\Delta(xy) = \Delta(x)\Delta(y)$, follows from the Mackey's decomposition theorem specialized to induction and restriction between products of symmetric groups.

Deep generalizations of this construction were developed by A. Zelevinsky [52], who constructed a similar biring structure on the sum (over all n) of Grothendieck groups of representation categories of $GL(n, \mathbb{F}_q)$, with parabolic induction and restriction in place of the usual induction and restriction. Many other examples of a biring structure on Grothendieck groups can be found in Bergeron and Li [4] and references therein.

In another paper, Zelevinsky classified irreducible representations of the affine Hecke algebra of S_n in terms of multisegments [53]. The significance of his result became clear much later, when S. Ariki [1] categorified all integrable irreducible representations of \mathfrak{sl}_k and affine \mathfrak{sl}_k via blocks of representation categories of cyclotomic quotients of affine Hecke algebra (an important earlier milestone was the work of Lascoux-Leclerc-Thibon on categorification of level one representations of affine \mathfrak{sl}_k via the representation categories of Hecke algebras of S_n at k -th root of unity, sum over all $n \geq 0$). I. Grojnowski [18], armed with ideas of A. Kleshchev from modular representation theory, gave an alternative derivation of Ariki's results.

Ariki also showed that Grothendieck groups of completions of affine Hecke algebra at suitable central ideals are canonically isomorphic to weight spaces of $U^+(\mathfrak{sl}_k)$ and $U^+(\hat{\mathfrak{sl}}_k)$, giving a conceptual categorification-style explanation for Zelevinsky's classification of irreducibles and proved that the basis of indecomposable projectives in these Grothendieck groups coincides with the $q = 1$ specialization of the canonical basis in the quantum group U_q^+ .

In Lusztig's construction of the canonical basis [35, 36], partially inspired by Ringel's work [42], canonical basis elements correspond to simple perverse sheaves on the varieties of quiver representations. In fact, his construction can even be viewed as a categorification of U_q^+ if one passes from the set of simple perverse sheaves to a suitable triangulated category of equivariant sheaves that these simple objects generate. Lusztig [35] and Kashiwara [23] also provided more elementary approaches to the canonical basis.

We'll now review a very down-to-earth categorification of U_q^+ based on a collection of certain rings, following [26], see also [44]. Grothendieck groups of these rings can be identified with an integral version of U_q^+ , while induction and restriction functors between these rings correspond to multiplication and comultiplication in the quantum group. A direct link between this categorification of U_q^+ and Lusztig's perverse sheaves construction was found by Varagnolo and Vasserot [47]. Brundan and Kleshchev [7] related these rings to Ariki's categorifications of U^+ and highest weight representations. A direct relation to derived categories of coherent sheaves on Nakajima quiver

varieties [41] is expected [10].

Let Γ be an unoriented graph without loop and multiple edges, with set of vertices I . The quantum group $U^+ = U^+(\Gamma)$ is a $\mathbb{Z}[I]$ -graded $\mathbb{Q}(q)$ -algebra with generators $E_i, i \in I$ of degree i and defining relations

$$E_i E_j = E_j E_i \text{ if } (i, j) \text{ is not an edge,} \quad (33)$$

$$(q + q^{-1}) E_i E_j E_i = E_i^2 E_j + E_j E_i^2 \text{ if } (i, j) \text{ is an edge.} \quad (34)$$

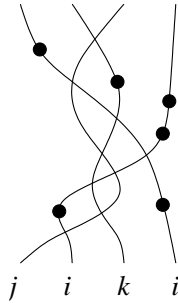
The integral version $U_{\mathbb{Z}}^+$ is the $\mathbb{Z}[q, q^{-1}]$ -subalgebra of U^+ generated by divided powers $E_i^{(n)} := \frac{E_i^n}{[n]!}$ over all i and n .

Both U^+ and $U_{\mathbb{Z}}^+$ are twisted bialgebras (the Cartan subalgebra is missing), with $\Delta(E_i) = E_i \otimes 1 + 1 \otimes E_i$ (instead of the usual comultiplication $\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i$ in quantum groups), and nonstandard algebra structure on $U^+ \otimes U^+$:

$$(x_1 \otimes x_2)(x'_1 \otimes x'_2) = q^{-|x_2| \cdot |x'_1|} x_1 x'_1 \otimes x_2 x'_2 \quad (35)$$

where $|x|$ is the degree, taking values in $\mathbb{Z}[I]$, and \cdot is the inner product on $\mathbb{Z}[I]$ with $i \cdot i = 2$, $i \cdot j = -1$ if i and j are connected by an edge, and $i \cdot j = 0$ otherwise. Relative to this algebra structure, Δ is a homomorphism $U^+ \rightarrow U^+ \otimes U^+$ (see [36] for more details).

Let $\nu \in \mathbb{N}[I]$, $\nu = \sum_{i \in I} \nu_i \cdot i$, $\nu_i \in \mathbb{N}$. For each such ν define the graded ring $R(\nu)$, as ring spanned by diagrams of lines in $\mathbb{R} \times [0, 1]$, with ν_i lines colored by label $i \in I$. Lines can intersect, but triple intersections are not allowed, neither are U-turns that create critical points under projection onto the y -axis. Lines may carry dots (in the picture below $\nu = 2i + j + k$).



Product is given by concatenation of diagrams; if the labels of endpoints don't match the product is zero. We allow isotopies (that do not create critical points relative to the y -axis projection) and impose the following relations.

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \begin{array}{c} i \\ i \end{array} = 0 \qquad \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \begin{array}{c} i \\ j \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} i \\ i \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} j \\ j \end{array} \quad \text{if } i \cdot j = 0$$

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \begin{array}{c} i \\ j \end{array} = \begin{array}{c} | \\ \bullet \\ | \end{array} \begin{array}{c} i \\ i \end{array} \begin{array}{c} | \\ | \end{array} \begin{array}{c} j \\ j \end{array} + \begin{array}{c} | \\ | \end{array} \begin{array}{c} i \\ i \end{array} \begin{array}{c} | \\ \bullet \\ | \end{array} \begin{array}{c} j \\ j \end{array} \quad \text{if } i \cdot j = -1$$

$$\begin{array}{c} \bullet \\ \diagup \diagdown \\ \diagdown \diagup \end{array} \begin{array}{c} i \\ j \end{array} = \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \begin{array}{c} i \\ j \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \qquad \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \begin{array}{c} i \\ j \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} = \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \begin{array}{c} i \\ j \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \quad \text{if } i \neq j$$

$$\begin{array}{c} \bullet \\ \diagup \diagdown \\ \diagdown \diagup \end{array} \begin{array}{c} i \\ i \end{array} - \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \begin{array}{c} i \\ i \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} i \\ i \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} i \\ i \end{array}$$

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \begin{array}{c} i \\ i \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} - \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \begin{array}{c} i \\ i \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} i \\ i \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} i \\ i \end{array}$$

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \begin{array}{c} i \\ j \\ k \end{array} = \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \begin{array}{c} i \\ j \\ k \end{array} \quad \text{unless } i = k \text{ and } i \cdot j = -1$$

The diagram shows an equation between three terms. The first term is a crossing of two lines, with the bottom-left strand labeled i , the bottom-right strand labeled j , and the top-right strand labeled i . The second term is a crossing of two lines, with the bottom-left strand labeled i , the bottom-right strand labeled j , and the top-left strand labeled i . These two terms are subtracted, as indicated by a minus sign between them. The result is equal to a third term, which consists of three parallel vertical lines. The bottom-left line is labeled i , the bottom-middle line is labeled j , and the bottom-right line is labeled i . To the right of this equation is the text "if $i \cdot j = -1$ ".

Make $R(\nu)$ graded by assigning degree 2 to a dot and degree $-i \cdot j$ to the (i, j) -crossing. Also, choose a base field \mathbb{k} and consider $R(\nu)$ as a graded \mathbb{k} -algebra. Diagrammatics make it clear that, at the cost of adding simpler diagrams, all dots can be moved above all crossings. Moreover, if two lines intersect twice in a diagram, the diagram can be further simplified. This argument gives a spanning set in $R(\nu)$ which consists of all monomials in dots times minimal length presentations of symmetric group elements as products of crossings, with arbitrary ν -colorings of the lines. One can check that this spanning set is a basis, by looking at the action of $R(\nu)$ on a suitable representation.

Each sequence $\underline{i} = i_1 \dots i_m$ of weight ν gives rise to an idempotent $1_{\underline{i}}$ in R_ν given by the diagram with m vertical lines labelled i_1, \dots, i_m from left to right. These idempotents are mutually orthogonal and $1 = \sum_{\underline{i} \in \text{Seq}(\nu)} 1_{\underline{i}}$, where $\text{Seq}(\nu)$ is the set of sequences of weight ν . Also, each idempotent $1_{\underline{i}}$ determines a graded projective module $P_{\underline{i}} = R(\nu)1_{\underline{i}}$.

Proposition 2 [26, 44] *There are natural isomorphisms of graded projectives*

$$\begin{aligned} P_{ij} &\cong P_{ji} \quad \text{if } i \cdot j = 0, \\ P_{iji}\{1\} \oplus P_{iji}\{-1\} &\cong P_{iij} \oplus P_{jii} \quad \text{if } i \cdot j = -1. \end{aligned}$$

This is a crucial proposition. After passage to the Grothendieck group, these isomorphisms become equalities on symbols of projectives:

$$\begin{aligned} [P_{ij}] &= [P_{ji}] \quad \text{if } i \cdot j = 0, \\ q[P_{iji}] + q^{-1}[P_{iji}] &= [P_{iij}] + [P_{jii}] \quad \text{if } i \cdot j = -1, \end{aligned}$$

showing a perfect match with the defining relations (33), (34) in U^+ after converting $[P_{ij}]$ to $E_i E_j$, $[P_{iji}]$ to $E_i E_j E_i$, etc.

The second isomorphism can be refined. Specifically, $R(mi)$ (m strands identically labelled) is naturally isomorphic to the nilHecke algebra, generated by multiplication by monomials x_1, \dots, x_m and divided difference operators ∂_i . It is well-known that the nilHecke algebra is isomorphic to the algebra of $m! \times m!$ -matrices with coefficients in the ring of symmetric functions in x_1, \dots, x_m . Consequently, the free module $R(mi)$ decomposes as a sum of $m!$ copies of an indecomposable projective module (the column module) which we denote by $P_{i(m)}$. The latter has a suitably selected overall grading shift, so that, taking grading into account, $R(mi)$ is the sum of $[m]!$ copies of $P_{i(m)}$.

When $m = 2$, there is a decomposition $P_{ii} \cong P_{i(2)}\{1\} \oplus P_{i(2)}\{-1\}$, and the second isomorphism above simplifies to

$$P_{iji} \cong P_{i(2)j} \oplus P_{ji(2)}.$$

Just like with the group algebras of symmetric groups, we can fit rings $R(\nu)$ over all $\nu \in \mathbb{N}[I]$ into a tower of algebras. There are natural (nonunital) inclusions $R(\nu) \otimes R(\nu') \subset R(\nu + \nu')$ described by placing diagrams representing elements of $R(\nu)$ and $R(\nu')$ in parallel, next to each other. Consider associated induction and restriction functors $\text{Ind}_{\nu, \nu'}$, $\text{Res}_{\nu, \nu'}$. We would like to look at the maps these functors induce on the Grothendieck group. The induction functor always takes projectives to projectives and induces a homomorphism between Grothendieck groups of finitely-generated graded projective modules

$$[\text{Ind}_{\nu, \nu'}] : K_0(R(\nu)) \otimes_{\mathbb{Z}[q, q^{-1}]} K_0(R(\nu')) \longrightarrow K_0(R(\nu) \otimes_{\mathbb{k}} R(\nu')) \longrightarrow K_0(R(\nu + \nu')).$$

The first arrow comes from tensoring projectives. In our case, the first arrow is an isomorphism, due to absolute irreducibility of simple graded $R(\nu)$ -modules, which we prove using methods of Kleshchev and Grojnowski. The second arrow comes from the induction functor.

For a general inclusion of rings, restriction functor does not necessarily take projectives to projectives. However, $R(\nu + \nu')$ (more accurately, $(1_\nu \otimes 1_{\nu'})R(\nu + \nu')$) is a projective $R(\nu) \otimes R(\nu')$ -module, so that the restriction does take projectives to projectives and induces a homomorphism

$$[\text{Res}_{\nu, \nu'}] : K_0(R(\nu + \nu')) \longrightarrow K_0(R(\nu) \otimes_{\mathbb{k}} R(\nu')) \cong K_0(R(\nu)) \otimes_{\mathbb{Z}[q, q^{-1}]} K_0(R(\nu')).$$

Form the direct sum

$$K_0(R) := \bigoplus_{\nu \in \mathbb{N}[I]} K_0(R(\nu))$$

and the corresponding sums of functors

$$\text{Ind} := \bigoplus_{\nu, \nu'} \text{Ind}_{\nu, \nu'}, \quad \text{Res} := \bigoplus_{\nu, \nu'} \text{Res}_{\nu, \nu'}.$$

These induce homomorphisms of K_0 -groups

$$[\text{Ind}] : K_0(R) \otimes K_0(R) \longrightarrow K_0(R), \quad [\text{Res}] : K_0(R) \longrightarrow K_0(R) \otimes K_0(R).$$

Theorem 1 [26] *There is a canonical isomorphism of twisted bialgebras over $\mathbb{Z}[q, q^{-1}]$*

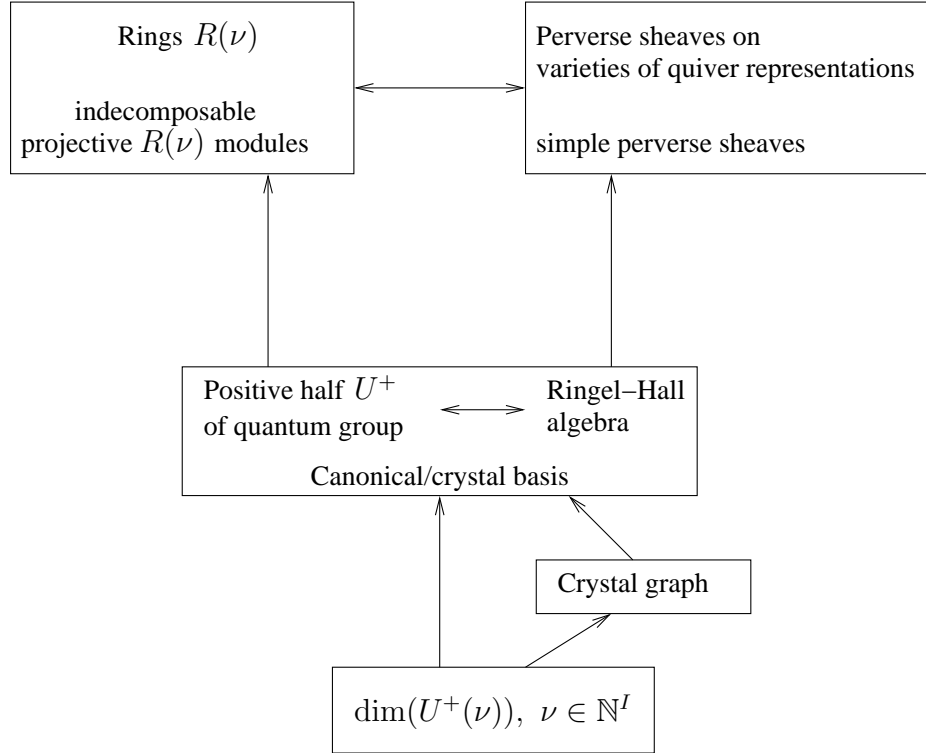
$$U_{\mathbb{Z}}^+ \cong K_0(R)$$

under which multiplication, respectively comultiplication, in $U_{\mathbb{Z}}^+$ is given by $[\text{Ind}]$, respectively $[\text{Res}]$.

This isomorphism takes the product element $E_{\underline{i}} = E_{i_1} \dots E_{i_m}$ to the symbol $[P_{\underline{i}}]$ of $P_{\underline{i}} = P_{i_1 \dots i_m}$ and divided power element $E_i^{(m)}$ to the symbol of $P_{i(m)}$, which is an indecomposable direct summand of P_{i^m} with a suitably normalized grading.

It is shown by Brundan and Kleshchev [8] (in the $sl(k)$ and affine $sl(k)$ case) and Varagnolo and Vasserot [47] (for general graphs Γ) that, if $\text{char}(\mathbb{k}) = 0$, the canonical basis [35, 23, 36] of $U_{\mathbb{Z}}^+$ goes to the basis of indecomposable projectives under this isomorphism. To get this match, when Γ has an odd length cycle, the definition of $R(\nu)$ should be slightly modified, see the above references and [27, 44]. Rings $R(\nu)$ proved handy in recent constructions [7, 9, 2] of graded versions of the group algebra of S_n , Specht modules, and q -Schur algebras, also see [20] and references therein for related developments. Moreover, they serve as a building block in Webster's categorification of Reshetikhin-Turaev link and tangle invariants [49, 50].

U^+ , its categorification, and related structures can be incorporated into the following (rather incomplete) diagram, where vertical up arrows denote categorifications.



At the base lies the collection of positive integers $\dim(U^+(\nu))$ – these are dimensions of weight spaces $U^+(\nu)$ of U^+ , or, more implicitly, coefficients of the Kostant partition function. Its categorification is the twisted bialgebra U^+ equipped with the Lusztig canonical/Kashiwara crystal basis. After the second round of categorification, the weight spaces $U^+(\nu)$ become Grothendieck groups of graded rings $R(\nu)$, and the canonical basis elements lift to indecomposable projective modules. Equivalently, one

can work with the derived category of equivariant constructible sheaves on varieties of quiver representations, with simple perverse sheaves being analogues of indecomposable projective graded $R(\nu)$ -modules. The upper horizontal arrow is a derived equivalence (for a carefully chosen version of the sheaves category) exchanging indecomposable projective $R(\nu)$ -modules and simple perverse sheaves.

Kashiwara crystal graph, together with Kashiwara operators, lies midway between the middle and base levels. Vertices of the graph correspond to canonical basis elements, and Kashiwara operators remember only “highest terms” for the action of E_i ’s on the canonical basis. This structure is set-theoretic. Despite its position somewhat below the first categorification, it is already incredibly rich. For instance, key results about simple and projective $R(\nu)$ -modules [26] follow via Kleshchev-Grojnowski constructions that amount to the representation-theoretical counterpart of the crystal graph structure.

U^+ is “one-half” of the entire quantum group $U_q(\mathfrak{g})$. In the categorification of U^+ the diagrammatics have a braid-like behaviour, in the sense that the lines in diagrams only go up and do not have U-turns. Interestingly, it is possible to enlarge the calculus of rings $R(\nu)$ by allowing U-turns, coloring regions of resulting diagrams by integral weights of $\mathfrak{g} = \mathfrak{sl}(n)$ and adding more relations, including all isotopies, to categorify the entire quantum group, see Lauda [32] for the $\mathfrak{sl}(2)$ case, [28] for arbitrary n , and Chuang-Rouquier [13], Rouquier [44] for related and less rigid axiomatics. The quantum group first needs to be modified, following Beilinson-Lusztig-MacPherson and Lusztig [36], by adding idempotents 1_λ of projection onto integral weights λ . In this categorification generators E_i and F_i become biadjoint functors (biadjoint up to grading shifts). The Grothendieck ring of the resulting 2-category is isomorphic to the BLM form of the quantum $\mathfrak{sl}(n)$. The diagrammatics of biadjoint functors described at the beginning of this paper plays a fundamental role in the definition of the 2-category and computations in it. Recent categorification of the q -Schur algebra by Mackaay, Stošić, and Vaz [37] links the Soergel category and categorified quantum $\mathfrak{sl}(n)$.

The $n = 2$ case, due to Lauda [32], settles a conjecture of Igor Frenkel, circa 1994, that there exists a categorification of quantum $\mathfrak{sl}(2)$ (some motivations for the conjecture can be found in [11]). Back in the early 90’s Igor Frenkel, who was the author’s advisor in graduate school, envisioned categorification lifting the entire theory of quantum groups, quantum 3-manifold invariants, and conformal field theory. Through the work of many people, his prophecy is becoming a reality.

Acknowledgments The author would like to thank Hiraku Nakajima and the Mathematical Society of Japan for the opportunity to deliver the Takagi Lectures in June 2009 and for their hospitality. The author is grateful to Aaron Lauda, Radmila Sazdanović, and Joshua Sussan for reading the manuscript and providing valuable comments and corrections. Partial support during the writing of present notes came from the NSF, via grant DMS-0706924. The list of references is grossly inadequate and misses quite a few fundamental contributions - our apologies for the omissions go to many fine practitioners in these areas.

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